**Example**

What angle/speed should throw football so that it hits a spot at certain time?



**Problem 1.**

Suppose you drop a bullet from height h = 1.5m. How long will it take to hit the floor? Would this time be different if you fired the bullet horizontally at speed v = 300m/s?

Acceleration due to gravity is ay = -g. So we have: y = y0 + v0yt + (1/2)ayt2 → 0 = 1.5 + 0(t) + (1/2)(-9.8)t2 → -1.5 = -4.9t2 → t = √(1.5/4.9) = 0.55s. And t would be same if you fire the bullet horizontally at any speed because y, y­0, v0y, ay would all still be the same, and so t would be the same. Basically, an object’s motion in one direction (say x), is independent of its motion in a different direction (say y).

**Problem 2.**

If you throw a ball straight up, and it takes 3s to come back down, how fast did you throw it?

We can use y = y0 + v0yt + (1/2)ayt2 again. The final and initial positions are 0, ay = -g = -9.8m/s2 and v0y is unknown. So… 0 = 0 + v0y(3) + (1/2)(-9.8)(3)2 → 0 = 3v0y – 44.1 → v0y = 44.1/3 = 14.7m/s (approximately 30mph).

**Problem 3.**

If you throw a football with speed v = 30m/s at an angle of 40° with respect to the horizontal, how high and far will it go?

To figure out the height we have the usual two y-equations: y = y0 + v0yt + (1/2)ayt2 and vy = v0y + ayt. The first reads: h = 0 + 30sin(40)t + (1/2)(-9.8)t2. But we don’t have t so we cannot yet get h. To get t we use fact that at the top of the trajectory, the football’s y-velocity will be 0. So we will have vy = v0y + ayt → 0 = 30sin(40) + (-9.8)t → t = 30sin(40)/9.8 = 1.97s. Now plug this into the h equation to get

h = 30sin(40)(1.97) +(1/2)(-9.8)(1.97)2 = 19m.

Now to get the distance it travels we will use the x eqn: x = x0 + v0xt + (1/2)axt2 → d = 0 + 30cos(40)t + (1/2)(0)t2 → d = 30cos(40)t. To get t, the time it takes to come back to the ground, we can either double the time we found above, or more generally, just use the y-eqn. Then we have: y = y0 + v0yt + (1/2)ayt2 → 0 = 0 + 30sin(40)t + (1/2)(-9.8)t2 → 0 = 30sin(40) + (1/2)(-9.8)t → t = 2∙30sin(40)/9.8 = 3.94s. Plugging this t into our x eqn, we get: d = 30cos(40)(3.94) = 90.5m.

3. A parachutist bails out and freely falls 50 m. Then the parachute opens, and thereafter she decelerates at 3 m/s2. She reaches the ground with a speed of 2 m/s. How long is the parachutist in the air?

The first 50m takes,



This gives her a velocity of:



Thereafter she accelerates at rate 3m/s2 and reaches final velocity of -2m/s. This would take a time,



So total time is t = 9.8 + 3.2 = 13s.

6. Fred throws a baseball upwards from an initial height of 2m with an initial speed of 10 m/s. At the same instant, directly above, Bob drops another baseball from the top of a 10m tall building. At what height will the two objects meet?

The position of the upward moving baseball as a function of time is given by,



The position of the falling baseball is:



They’ll be at the same height when,



and their height at that time will be:



4a. Suppose you throw a baseball upward, releasing it from a height of 2m off of the ground with initial velocity 28m/s. Write down an expression for its velocity vy and position y, as a function of time.

We have:



4b. What will be its height and velocity when t = 5s?



4c. When will it reach its highest position, and what will that be?

It will reach highest position when vy = 0, which is when,



and height at this time will be:



4d. When will it have fallen to back its original position.

It will be back when,



we’ll note that this is twice the time it takes to get to the top, as it should be.

5a. Suppose you throw a baseball from initial position x0 = y0 = 0, with an initial velocity **v**0 = 35m/s @ 20° above the horizontal. Write down an expression for vx, vy, x, and y as a function of time.

First we must break the velocity vector into its components. In this case, we will have **v0** = (35cos20°, 35sin20°) = (33,12). Therefore v0x = 33, and v0y = 12. Now we can write down the equations,



5b. How high will the baseball go?

To figure out the max height, we must first determine when it will reach that height. It will reach that height when vy = 0, i.e. when 12 – 9.8t = 0, when t = 12/9.8 = 1.22s. Plugging this time into the y equation we have y = 12(1.22) – 4.9(1.22)2 = 7.35m.

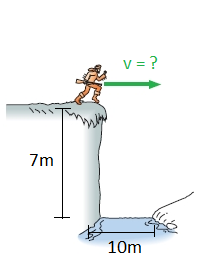
5c. When will it hit the ground?

It will hit the ground when y = 0, i.e. when 12t – 4.9t2 = 0, when t(12 – 4.9t) = 0, when 12 – 4.9t = 0, when t = 12/4.9 = 2.44s.

5d. How far away will it land?

And it will be at horizontal position x = 33t = 33(2.44) = 80.5m.

**Question 4**. How fast would one have to run off of a 7m cliff to land 10m away? Note drawing isn’t exactly to scale.



Filling our information into the x equation we have:



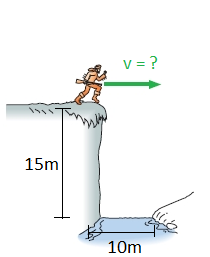
and filling it into the y-equation we have:



and plugging this back into the x-equation we get:



**Question 3**. How fast would one have to run off of a 15m cliff to land 10m away?



Filling our information into the x equation we have:



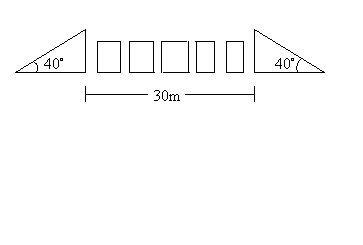
and filling it into the y-equation we have:



and plugging this back into the x-equation we get:



7. A motorcyclist rides up a ramp angled at 40 degrees with respect to the ground to jump over a row of cars 30m long and land on a ramp on the other side (also angled at 40 degrees). How fast should he be going at the top of the ramp in order to succesfully complete the jump?



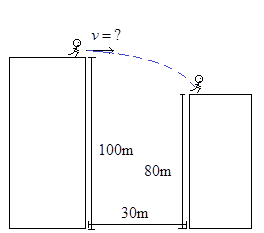
We can use the range equation since he begins and ends at the same height,



Solving for v0 we have.



6a. Suppose you’re Trinity, escaping from agent Smith by running off the top of a tall building and landing on the one below it. We want to know how fast you must run off the ledge to successfully land on the smaller building. To that end, write down expressions for x, y, vx, and vy as a function of time leaving any variables you don’t know as symbols.



First we must vectorize her initial velocity. It is **v**0 = (v,0). So v0x = v, and v0y = 0. Now we can write down the equations



6b. Determine when she reaches the lower building and her required velocity by setting her x and y expressions equal to the appropriate values and solving the resulting equations.

She’ll reach the building when,

 and 

Plugging this time into the v = 30/t equation we have:

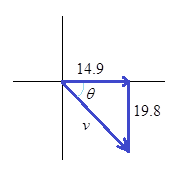


6c. Just for curiosity’s sake, use the time you found above, along with the vx and vy equations to compute the magnitude and direction of her velocity upon landing on the top of the lower building.

To determine her velocity when she hits the lower building we will plug t = 2.02s into her velocity equations.

 and 

Plotting this vector we have:



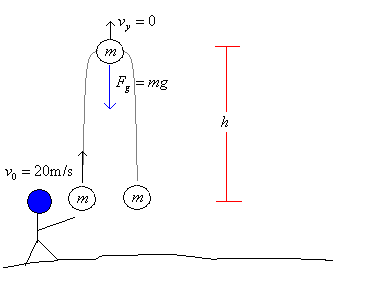
and its magnitude/direction are:



**Example: 1D free fall**

Suppose you throw a ball upwards with initial velocity of 20m/s (≈ 44mph). How high will it go? How long will it take to come back down?

To proceed, we draw a picture of the process with all the relevant information. We show it starting out with velocity v0 = 20m/s, upwards, and with the force Fg = mg acting on it. The force acts on it everywhere, of course (even when it isn’t in the air), but we’re only displaying the force at the top. We’re also displaying the fact that at the top of the trajectory, vy = 0 – this is generally true since if it were not 0, then it would keep going up and then it would not be at the top of the trajectory.



First we use N2L to determine the acceleration. All the forces are in the y-direction so we just have,



which we already knew. Now since we want to know how high it will go, and we have information about the velocities at the beginning and end of h, it is convenient to use the following formula,



applying this to our situation,



So it will rise to a height of 20.5m. As far as when it will return, let’s look at the equation for the position of the ball, what this is equivalent to asking is, when will y = 0 again. So to answer, let’s look at the y-equation of motion.



Set y0 = 0, v0y = 20m/s, and ay = -9.8m/s2. And then solve for when y = 0.



6. A car ball rolls off a horizontal cliff with an initial speed of 30 m/s, and falls 40 m into a lake below. Determine when the car hits the lake.

Again, we can use the equation:

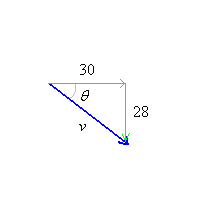


7. In the problem above, what is the magnitude *and direction* of the car’s velocity just before it strikes the water?

The velocity of the car is given by:



If we draw this vector, it will look like:



So then the magnitude and direction of the velocity of the car is:



5. A golf ball rolls off a horizontal cliff with an initial speed of 15 m/s. The ball falls a vertical distance of 20 m into a lake below. What is the speed *v* of the ball just before it strikes the water?

First we need to figure out when the ball hits the water. We can do this with the y-equation…



Now use the vx and vy equations to get the components of the velocity when it hits the lake. First vx…



and now vy…



So the magnitude of the velocity is:



**P6.** Suppose you are 36m away from the goal post (which is 3m high). How fast must you kick the ball (at a 37˚ degree angle) so that it will top the goal post?

Again we can examine the position of the object as a function of time,



When the ball is 36m away, we want it to be 3m high (so it will top the goal post). So we require x = 36, and y = 3. Doing this to the x-coordinate we have,



We don’t know t, but if we plug this into our equation for y, then we can get t. So setting y = 3 and plugging this expression for v0 into it gives,



So we see that the football will hit the goal post 2.2s later. We can now use this time for the velocity. Remember that we solved for v0 to get,



and so



So we must kick the ball to a speed of 20.5m/s.

**P7.** What is the maximum height to which the football will rise?

There are a few ways to do this. One is...we can determine we can examine the velocity of the football in the y-direction.



Then we’ll note that it is when vy = 0 that it will be at the top of its trajectory. So we find out when this is, and then plug that time into the y-coordinate of the position function. So first solving for vy = 0, we have,



and then plugging into the y-coordinate of the position function,



and so we’ll have,



Another way is to use the equation,



Let the initial position be the ground, when kicked, and the final position be when its at the top of its trajectory. Then we have,



**Question 2.** A model rocket’s engine provides a thrust, F = 12N, for 5s before running out of fuel. The 0.450kg rocket continues to rise upward eventually reaching what height?

While the engine is firing we have:



which gives it a velocity and altitude of:



before the engine shuts off. Next the rocket will continue upwards with this as its initial velocity and altitude until comes back down. This will happen when:



and its height at this time will be:



6. A hot-air balloon is rising upward with a constant speed of 5 m/s. When the balloon is 7.1 m above the ground, the balloonist accidentally drops a compass over the side of the balloon. How much time elapses before the compass hits the ground?

We can find the time with the y-equation…



Now use quadratic equation…



4. You are in a hot-air balloon, ascending at a rate of 10m/s, at a height of 100m. If you throw a rock horizontally off of the balloon, how long will it take to hit the ground?

Use the y equation of motion. We have:



**Question 7**. You are in a hot-air balloon, ascending at a rate of 10m/s, at a height of 100m. If you throw a rock horizontally off of the balloon with a speed v = 35m/s, how far will it go horizontally before hitting the ground?



Solving for t…

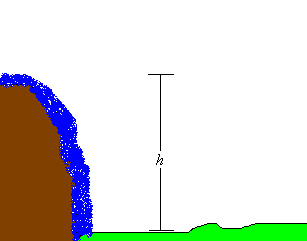


Then filling this into the x equation:



**Example: Height of a waterfall**

Suppose we want to calculate the height of a waterfall. All we have to do is time how long it takes for one of the water droplets (we can follow it with our eyes) to hit the ground. Suppose it takes 2.4s. Then how high is the waterfall?



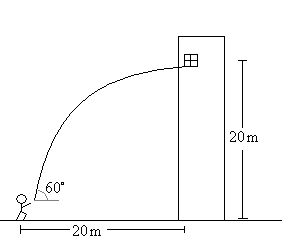
The velocity of the water coming off of the top of the waterfall is unknown in the x-direction (the velocity would be how fast the river is moving at the top of the waterfall), but in the y-direction it is initially 0, since it is initially going horizontally. Therefore, just concentrating on the y-component of motion we ask ourselves how high must the water fall from in order to hit the ground in 2.4s, given an initial v0y = 0. To answer that question, first we solve for the acceleration of the water droplet, using N2L,



as always for free fall. And then we plug it into the y equation of motion. Let y0 = 0 at the top of the waterfall. So then,



**Question 3**. Juan is locked in the Bastille. Fortunately for him, Joe has a copy of the jail keys in his posession. Suppose that Juan is locked in a cell 20m above the street, and that Starbuck can get no closer than 20m from the tower base due to the presence of French guards. How fast should he throw the keys to Juan in order for him to catch them, if he angles his throw at 60 degrees above the horizontal? Note: the pictured trajectory may not be entirely accurate, but it gives an idea.



We use the equation, taking the initial position to be 0.



and we want this to end up at the position,



So we equate,

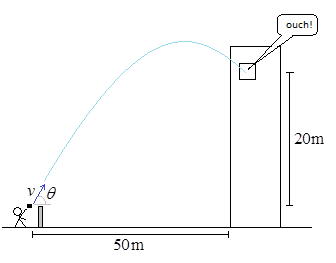


Now solving this equation for v0…



**Problem 2**

Rapunzel is locked away in the Bastille, but you have keys to her prison. Unfortunately, you cannot get any closer than 50m from the tower, and her cell is at height of 20m above the ground. If you throw the keys at an angle θ = 35°, then at what speed v must you throw them so that they land in her cell? As according to the diagram, you may take the initial and final position of keys to be (x0 = 0, y0 = 0), (x = 50, y = 20) respectively. Note you will have to solve two equations with two unknowns to get v.



Equations are:



And now plug this into the y-equation:

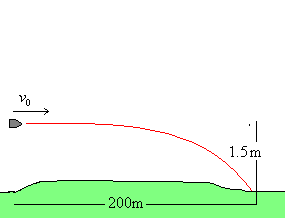


Now plug this into the velocity equation to get



**Example: Speed of a bullet**

What is one way to measure the muzzle velocity of a bullet? Suppose you fire a rifle horizontally, from a height of 1.5m, and the projectile lands 200m away. What was the initial speed of the bullet?



First, from N2L, the acceleration of the bullet will be ay = -9.8m/s2 since it is in free fall between the time it is fired and the time it hits the ground. Now to determine its initial speed, since we know the x displacement we might want to use the equation,



Let x0 = 0, and x = 200. ax = 0 since the acceleration is purely in the y direction and so we have,



So if we knew how long it took to hit the ground, we could determine v0. To get that information we look at the y displacement. Use the analogous equation,



Let the initial position be 1.5m, then y = 0. v0y = 0 because initially the bullet is traveling horizontally. Finally ay = -9.8m/s2 of course. Filling all this in we have,

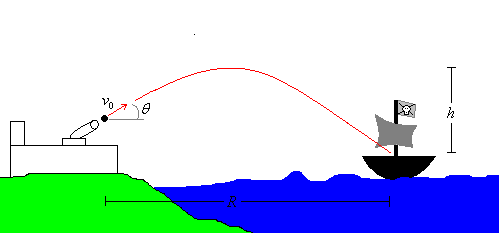


So filling this information into the x-equation givs us:



**Example: Firing a projectile**

Suppose we’re defending a fort from pirate ships. We want to fire our canon balls at the requisite angle to hit the pirate ships below. If the muzzle velocity of cannon balls is v0 = 100m/s, the ship is a distance R = 800m away, then at what angle should the cannon be aimed? (we’ll neglect the difference in height between where the cannon is fired and the deck of the pirate ship). To what height will the projectile rise?



The answer is easy to formulate. From N2L, since the projectile will be in free fall during its flight, its acceleration will be ay = -9.8m/s2. Now in order to determine R we need the equation which gives us the x-coordinate. This is:



Problem is, we don’t know t, i.e., we don’t know when the projectile reaches the distance x = R. To find t, we go to the y-equation, like in the previous problem.



Now we recognize that when the projectile reaches the distance x = R, y will be 0. So we substitute this in and solve for t (in terms of v0).



Now that we have t, we plug it back into the x equation, and see what we get,



The last equation is useful in its own right – it is called the range equation, which gives the distance a projectile travels when fired at a certain speed/angle.



Solving for θ, though, we get,



To find the height of the projectile, we use the fact that at the top of the trajectory vy = 0. And so apply the equation:

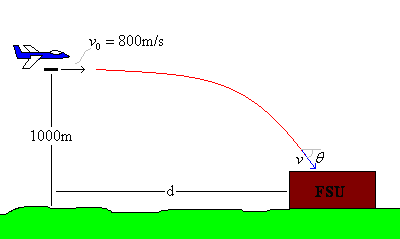


to it. So,



**Example: Firing a missile**

Consider this example. We’re flying an F-22 at an altitude of 1000m and speed 500m/s on a mission to destroy FSU. We fire a missile at a speed of 300m/s (relative to the plane). How far away should we be from FSU when we fire the missile? How long will it take for the missile to hit? And what will be its velocity when it does?



This time, I’ll do the example using vectors. We could have done the previous problems this way too. One can do it either way. First let’s look at the position of the missile as a function of time.

The x coordinate of the missile is given by:



The y coordinate is given by,



When it hits FSU, the y coordinate will be 0. So we can see when the missile will hit FSU. This will be when,



During that time t, it will travel the distance d. So we can determine what d is by inserting this time into the x-equation,



So we need to be 11 440m away (about 7.15 miles), and the missile will take 14.3s to hit. To get the velocity upon impact, we can use the equations



Now the initial velocity is (its velocity is purely in the horizontal direction). The acceleration vector is  (there is never any acceleration in the horizontal direction during free fall motion). So the velocity at 14.3s is:



So the velocity upon impact is:



and we can determine the magnitude and direction of the velocity,



A perhaps more concise way of doing this problem, and some of the other ones above, is to do it with vectors. So let’s solve the problem again. To determine the proper distance from which we should fire the missile, we will form the position vs. time function,



and we want it to hit FSU, so we need,



So we need to be 11 440m away (about 7.15 miles), and the missile will take 14.3s to hit. To get the velocity, we can take the derivative of **r**(t):



So the velocity at 14.3s is:



So,



**Question 6.** A supply plane needs to drop a package of food to scientists working on a glacier in Greenland. The plane flies 100m above the glacier at a speed of 180m/s . How far short of its target should it drop its package?

In the y-direction we have:



So in the x-direction we would have:



**Example: What’s the terminal velocity of a bowling ball, a penny?**

The terminal velocity for a 10kg bowling ball would be something like the following. m ≈ 10kg, C ≈ 0.5, ρ ≈ 1.2kg/m3, A ≈ π(0.10)2m2 = 0.0314m2. Filling these in, we get



or about 225mph. According to the web, for a penny, m = 3g, and the cross sectional radius is ~ 1cm. So…



so we can see that heavier things to tend to fall faster than lighter things, when air resistance is a factor. Note how this debunks the urban legend that a penny dropped off of the Empire State building will kill you. 12.5m/s is around 30mph – hardly enough to kill someone – though it might sting a little.

**Velocity as a function of time**

Let’s solve for the velocity as a function of time. Go back to:



to solve this non-linear differential equation we can do many things. One is the following:



For simplicity, let’s use,



then we can write this as:



We can use the method of partial fractions to evaluate this anti-derivative. That is, we note that:



and so,



exponentiating both sides to solve for v, we get,



Now supposing that our initial velocity is 0, we can solve for the arbitrary integration constant…



Thus, v(t) is given by:



We can write this in a neater fashion,



In the last lines we use the following definitions that I think you learn in Calc. 2.



So we have,



The tanh function looks like this:

**Position as a function of time**

If we want the position as a function of time, then we must integrate once more. Using the fact that the derivative of cosh(x) is sinh(x) (you can verify this to be true from the definitions above), this comes out to be,



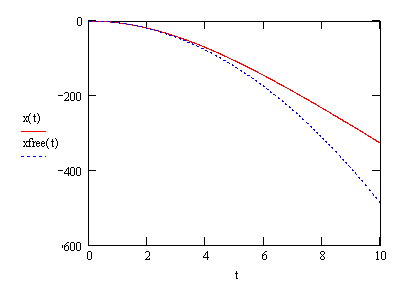
So, assuming we drop the ball from the reference height of 0, we have:



Compare this to the formula for x(t) when x0 = 0, and v0 = 0, ignoring air resistance,



Compare these two, assuming a terminal velocity of 50m/s.



You can see that for about 3s, the motion is basically the same – after which they begin to separate. After 10s, the ball will actually have fallen 300m, while our free fall approximation suggests it will have gone 500m.

**Example: Falling penny**

Suppose you drop a penny from the empire state building. How long will it take to hit the ground?

Well, the position as a function of time is given above. We just need vT, the terminal velocity of a penny. This is approximately, using C ≈ 1,



(pretty slow huh!). Therefore, to find the required time we simply solve the equation for t.



Plugging in the values we get,



The estimate neglecting air-resistance would be:



Clearly air resistance makes a big difference.

12. Suppose you drop a ball downward from a tall building. If the ball’s mass is m = 3kg, and it is subject to an air resistance force of magnitude F = 0.063 v2, what will be its terminal velocity?

We can write down N2L in the y-direction as:



Once the ball reaches terminal velocity, it no longer accelerates. So then ay = 0. Filling this and solving for vy we get:



**Example**

Suppose free fall parameter γ = 10. Mass = 75kg. What is vT? What v(t), y(t) if fall from building at height h?



and then,



Integrating again,

